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ON NONLOCAL DAMAGE MODELS FOR INTERFACE PROBLEMS

Z. CHEN†
New Mexico Engineering Research Institute, University of New Mexico, Albuquerque, NM 87131-1376, U.S.A.

and

H. L. SCHREYER
Department of Mechanical Engineering, University of New Mexico, Albuquerque, NM 87131-1376, U.S.A.

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Abstract—Consistent with thermodynamic restrictions, nonlocal formulations of damage models are investigated with regard to their analytical and numerical capabilities for predicting the evolution of microcracking. To trace the post-limit structural response with an efficient numerical algorithm, an incremental–iterative solution strategy is constructed through the use of an initial elasticity stiffness matrix and an evolving-localization constraint. The constraint is related to a suitable measure of microcracking at the most severely damaged element that changes with the evolution of a localization zone. As a simple phenomenological approach, a nonlocal feature is incorporated into the constitutive model through the gradient of a scalar measure of damage, and a symmetry boundary condition is invoked for the nonlocal governing differential equation. In numerical calculations, the gradient is evaluated through a simple difference scheme. For applications to the interface problems with geologic media, the proposed procedure is verified with analytical solutions for one-dimensional cases, and illustrated with two-dimensional cases for which experimental observations are available. Some important theoretical and computational issues associated with nonlocal models are then discussed based on the results obtained.

1. INTRODUCTION

The degradation of material properties accompanied by the evolution of a localized deformation zone, i.e. damage with localization, is a topic of great current interest in nonlinear structural analyses. A combined effort in experimental, analytical and computational aspects is being conducted worldwide to advance the relevant methodologies.

The use of both conventional and micro-experimental techniques, involving image analysis, X-ray radiography, laser holographic interferometry, and scanning electron microscopy, has produced a better understanding of the physical nature behind failure phenomena (Chen, 1989; Fishbine *et al.*, 1990; Maji *et al.*, 1990; Shah and Gopalaratnam, 1985; Van Mier, 1984; Wang *et al.*, 1991; Wawersik and Brace, 1971). Material failure of engineering structures arises from two distinct modes of microstructural changes: one is plastic flow and the other is the degradation of material properties. Plastic flow, which is reflected through permanent deformation, is the consequence of a dislocation process along preferred slip planes as in metals, or particle motion and rearrangement as in geologic materials. Because the number of bonds between material points is hardly altered during the plastic process, the material stiffness remains insensitive to this mode of microstructural motion, and change of strength is reflected through plastic strain hardening and apparent softening. On the other hand, the nucleation, crushing, and coalescence of microcracks and microvoids result in debonding, which is reflected through the degradation or damaging of material stiffness and strength. Both apparent softening and damage are often accompanied by the evolution of a localized deformation zone. In general, both modes are present and interacting, and a structure starts to fail when macrocracks form and propagate from the cluster of microcracks which defines the localization zone. Hence, nonlocal mechanisms, as manifested by the evolution of inhomogeneous interactions among material points, are

† To whom correspondence should be addressed.

essential features of failure phenomena. In other words, local constitutive models, in which the stress at a material point is related to the strain only at that point, might not be representative of failure mechanisms of engineering structures.

Many investigations have been performed to resolve theoretical and modeling issues associated with apparent softening and damage accompanied by localization, as reflected by the current mechanics literature. Among the continuum models proposed are nonlocal plasticity and damage models, rate-dependent models, Cosserat continuum models and micromechanical models. Preliminary results obtained for one- and two-dimensional sample problems look quite promising because, if a nonlocal model is used, the failure behavior of sample structures can be simulated and the results are mesh independent so that nonzero energy dissipation can be obtained in the post-limit regime. In order to predict the essential features of failure phenomena, several specific forms of nonlocal models have been suggested and include the use of gradients of strain (Chen and Schreyer, 1987; Triantafyllidis and Aifantis, 1986; Schreyer and Chen, 1986), imbricated continua (Bazant *et al.*, 1984), prescribed zones of localization (Pietruszczak and Mroz, 1981) and weighted integral averages of damage (Pijaudier-Cabot and Bazant, 1987). Although each approach has an appealing feature, there exists an insufficient number of analytical and numerical solutions to show if any one should be preferred, based on correlations with experimental evidence or on the ease with which analytical or numerical solutions can be obtained. Nevertheless, it can be shown that the field equations governing static softening responses are strongly elliptic, and therefore are well-posed, if a nonlocal strain–gradient model is employed (Chen, 1989). Since the order of derivatives of deformation appearing in the nonlocal governing equations is higher than the order normally appearing in local governing equations, additional boundary conditions are required around the localization zone if a well-posed solution is to be obtained to a physically unstable problem involving localization. With the assumption of a symmetry boundary condition, analytical solutions to the one-dimensional field equation have been achieved for a set of nonlocal plastic-strain–gradient models and various forms of nonlocal damage–gradient models, for which material parameters can be identified from available experimental data (Chen, 1993a; Schreyer, 1989, 1990). However, the physical and mathematical foundation for nonlocal models remains at an infant stage, especially, the physics behind the additional boundary conditions is still not clear.

From a computational point of view, there exists two major difficulties in predicting structural failure. One is the occurrence of an ill-conditioned tangent stiffness matrix around critical points, namely, limit and bifurcation points. The other is the selection of a suitable constraint on the solution path such that the post-critical response can be traced. Since a robust and efficient solution scheme is necessary to make failure simulation available in a routine manner, several procedures have been proposed to circumvent the difficulties associated with critical points (Belytschko and Fish, 1989; Chen, 1989; Chen and Schreyer, 1990a, 1991; De Borst, 1987). The standard arc-length control is still commonly employed in geometrically nonlinear cases, and with some modifications in materially nonlinear cases. Because the failure zone is localized into a small region as damage occurs, the arc-length constraint formulated in the global deformation field is insensitive to the evolution of the localized deformation mode. As a result, a suitable constraint should be constructed in terms of a localized kinematical field if damage with localization needs to be predicted. Due to the fact that the localization zone is evolving and there is a sign change of the load increment at a critical point, the control point (element) should also be taken to vary in position with a suitable measure of damage, and the localized constraint parameter that reflects the extent of irreversible damage should be constrained to increase monotonically. Preliminary results obtained for one- and two-dimensional sample problems indicate that the use of an evolving localized control is a reasonable choice for localization problems including snap-back or snap-through (Chen and Schreyer, 1990a, 1991). As a remedy to avoid the use of an ill-conditioned tangent stiffness matrix around critical points, a secant stiffness matrix based on continuum damage mechanics has been used with a dramatic increase in the rate of convergence with respect to the rate obtained using a tangent stiffness. It should be emphasized that the use of a secant stiffness matrix does not exclude the existence of critical points that must be identified by analyzing the corresponding tangent stiffness matrix. The theoretical insight associated with critical points can be explicitly obtained only from the tangent stiffness matrix, while the use of a secant stiffness matrix improves the convergent behavior of the iterative numerical technique. Recent research on localization and bifurcation suggests that a more efficient method might be available for predicting different orientations and resolving details of localized deformation patterns (De Borst *et al.*, 1993; Neilsen and Schreyer, 1992; Peric *et al.*, 1991; Pijaudier-Cabot and Benallal, 1993; Zbib and Jubran, 1992). For a general failure assessment, however, more research needs to be performed to yield a robust solution scheme that has a fast rate of convergence and follows the correct solution path with an efficient semi-discretization procedure.

As an extension of previous work, an attempt is made here to establish a basic framework of nonlocal damage models based on the internal variable theory of thermodynamics. Theoretical and computational aspects of the models are investigated with regard to their applicability to localization problems. Specifically, the damage consistency condition is explored in detail to clarify the nonlocal terms. Similar results hold for nonlocal plasticity models which are not covered in this paper for the purpose of conciseness. In order to increase computational efficiency, an incremental–iterative solution strategy is developed through the use of an initial elastic stiffness matrix and an evolving-localization constraint. The constraint is related to a suitable measure of damage at the most severely damaged point that varies in position with the evolution of a localization zone. The use of the initial elastic stiffness matrix implies that only one decomposition is necessary and rapid convergence is retained. For applications to interface problems with geologic media, the proposed procedure is verified with analytical solutions for one-dimensional cases, and illustrated with two-dimensional cases which replicate experimental observations. Finally, the limitation and possible improvement of existing nonlocal formulations are discussed based on the current work.

2. A BASIC FRAMEWORK OF DAMAGE MODELS

Phenomenological and micromechanical approaches are two methods of formulating constitutive models for nonlinear analyses. There is no question that a meaningful constitutive model must be consistent with the micromechanics of the material considered. However, a detailed discussion of micromechanical features is well beyond the scope of the paper, and instead, the restrictions of thermodynamics are imposed on the development of a basic phenomenological framework of damage models.

Much work has been done to develop damage models based on the internal variable theory of thermodynamics, with coupled or uncoupled elastoplastic-damage and small or finite deformation assumptions (Chen and Schreyer, 1990b; Ju, 1989a, b; Krajcinovic, 1989; Yazdani and Schreyer, 1988, 1990). In order to clarify the roles played by local and nonlocal internal (macroscopic) variables that represent the internal constitution of the material, an alternative procedure is outlined below. The equations are expressed in terms of direct notation with bold-faced letters denoting tensors of first or higher orders.

For a strain-based formulation, a general form of the internal energy per unit volume can be defined in terms of a suitable strain tensor, $\mathbf{e}$, and a set of internal variables, I_i

$$U = U(\mathbf{e}, I_i). \tag{1}$$

The number and specific forms of the internal variables depend on the model being considered, e.g. local or nonlocal, rate-independent or rate-dependent, and isotropic or anisotropic. Within the context of a purely mechanical case, the Clausius–Duhem inequality is given by

$$-\dot{U}(\mathbf{e}, I_i) + \mathbf{s} : \dot{\mathbf{e}} \geqslant 0 \tag{2}$$

for any admissible process, in which a superposed dot denotes the derivative with respect to a parameter, such as time, used to describe the loading process. The stress tensor $\mathbf{s}$ is conjugate to **e** based on the internal work per unit volume. From the definition of the internal energy, an alternative form of eqn (2) can be expressed as

$$\left(\mathbf{s}-\frac{\partial U}{\partial \mathbf{e}}\right):\dot{\mathbf{e}}-\sum_i \frac{\partial U}{\partial I_i}\dot{I}_i \geqslant 0. \tag{3}$$

If the strain rate is considered arbitrary and independent of the internal variables, then it follows that

$$\mathbf{s}=\frac{\partial U}{\partial \mathbf{e}} \tag{4}$$

and the remaining term in (3) yields the dissipative inequality

$$-\sum_i \frac{\partial U}{\partial I_i}\dot{I}_i \geqslant 0. \tag{5}$$

Up to this point, no statement has been made to indicate whether coupled or uncoupled cases and small or finite deformations are considered. In order to establish a specific constitutive model, certain reasonable assumptions are required.

For the purpose of simplicity, it is assumed that deformations are small, and that the effects of plasticity can be ignored which is often the case for brittle materials. Thus, no distinction needs to be made among different measures of stress and strain, and internal variables are related only to the damage process. Based on experimental observations, the specific form of the internal energy can then be postulated to be

$$D = \tfrac{1}{2}\mathbf{e}:\mathbf{D}^{ed}:\mathbf{e} \tag{6}$$

in which $\mathbf{D}^{ed}$ denotes a fourth-order secant stiffness tensor which is a function of damage internal variables and can be isotropic or anisotropic. Due to the functional form of the internal energy, $\mathbf{D}^{ed}$ possesses major and minor symmetries. The combination of eqns (4) and (6) yields the stress–strain relation

$$\mathbf{s}=\frac{\partial U}{\partial \mathbf{e}}=\mathbf{D}^{ed}:\mathbf{e}. \tag{7}$$

From the dissipative inequality (5) and equation (6), it follows that

$$-\tfrac{1}{2}\mathbf{e}:\dot{\mathbf{D}}^{ed}:\mathbf{e}\geqslant 0. \tag{8}$$

Since the damage process is irreversible, a set of evolution equations must be defined. A general form can be written as

$$\dot{\mathbf{D}}^{ed} = -\sum_i \dot{I}_i\mathbf{Q}_i \tag{9}$$

in which the fourth-order tensors $\mathbf{Q}_i$ may be functions of I_i. In order to identify the elastic and damage regime, consider a damage function $F(I_i)$. A damage surface function f is obtained by postulating

$$f(\mathbf{e},I_i)=\frac{1}{2}\sum_i \mathbf{e}:\mathbf{Q}_i:\mathbf{e}-F^2, \tag{10}$$

where $f=0$ yields an elastodamage surface. The formulation of eqn (10) identifies three regimes: a region inside the surface that defines elasticity, a region on the surface that defines damage and a region outside the surface that cannot be reached. If the $\mathbf{Q}_i$ are positive semi-definite with at least one positive definite, then the dissipative inequality is satisfied when $f = 0$ and damage is occurring ($\dot{I}_i \geqslant 0$). It should be pointed out that the theory of thermodynamics does not yield a specific constitutive model, and instead, only imposes certain constraints on the functional forms of the constitutive relations that are postulated. The elastodamage surface is usually assumed to be closed and convex. To determine the set of internal variables from a single surface, a suitable set of damage rules must be defined as follows:

$$\dot{I}_i = \dot{\lambda} R_i, \tag{11}$$

where R_i may be scalar functions of I_i, and λ is a monotonically increasing parameter used to parametrize the irreversible evolution of damage. For a given strain rate $\dot{\mathbf{e}}$, the damage consistency condition, $\dot{f} = 0$, is used to find $\dot{\lambda}$. The consistency condition is a direct consequence of the loading and unloading criteria given by

$$f \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad f\dot{\lambda} \equiv 0. \tag{12}$$

It can also be shown that the use of eqns (9) and (11) results in an alternative expression for the rate of damage tensor

$$\dot{\mathbf{D}}^{ed} = -\dot{\lambda} \sum_i R_i \mathbf{Q}_i = -\dot{\lambda} \mathbf{N} \quad \text{with} \quad \mathbf{N} = \sum_i R_i \mathbf{Q}_i, \tag{13}$$

where the fourth-order tensor $\mathbf{N}$ identifies the direction of damage. However, the explicit definition of the damage rules, eqn (11), clarifies the roles played by local and nonlocal internal variables, as shown later with examples.

The basic phenomenological framework for a strain-based formulation is complete at this point for damage models. The stress-based counterpart follows the use of a Gibbs free energy per unit volume, $G(\mathbf{s}, I_i)$, and the corresponding Clausius–Duhem inequality is given by

$$\dot{G}(\mathbf{s}, I_i) - \mathbf{e} : \dot{\mathbf{s}} \geqslant 0. \tag{14}$$

With the use of the stress–strain relation, the strain-based formulation can be changed to a stress-based one which is often the case for existing plasticity models.

Specific cases of common interests can be addressed within this framework, e.g. isotropic or anistropic damage, rate-independent or rate-dependent damage, and even damage deactivation (Hansen and Schreyer, 1992). In particular, the features of specific nonlocal formulations can be clearly identified, as discussed in the next section. For damage healing (Brodsky, 1990), namely, the decrease of the amount of damage during the loading process, it is believed that thermal and chemical effects must be included in the constitutive relations to satisfy thermodynamic restrictions, a topic which is beyond the scope of the paper. The difference between various models depends on the specific forms of evolution equations, elastodamage surfaces, and damage rules for internal variables. It is interesting to note that the formulation of the damage evolution laws is parallel with that of the plastic flow rules. In fact, there exists a similarity between the formulations of damage and plasticity models from a thermodynamic point of view (Chen and Schreyer, 1990b). Hansen and Schreyer (1993) have explored this parallelism to a considerable degree.

Theoretically speaking, the current state of stress and internal variables can be determined for a given state of strain according to this framework. Except for some extremely simple cases, however, a suitable incremental–iterative scheme must be invoked to solve the set of highly nonlinear equations. If a nonlocal model is of concern, the case becomes even more difficult, because not only a lower-order differential operator is involved as for local models but also an integro-differential or inhomogeneous differential operator might occur. With the split between local and nonlocal internal variables, the methods to circumvent this numerical problem are explored next.

3. NONLOCAL FORMULATIONS

3.1. *General representation*

As can be seen from the phenomenological framework of damage models, the set of internal variables can not be determined without knowing $\dot{\lambda}$ for the damage process. Hence, the solution procedure for this monotonically increasing parameter based on the damage consistency condition plays a crucial role in an incremental–iterative constitutive-model-solver. For a local model, a rootfinding numerical scheme such as the Newton–Raphson or secant method can be adopted to find an increment, $\partial\lambda$, for a given total strain increment (Chen and Schreyer, 1990b and 1991; Simo and Hughes, 1988). If a nonlocal model is considered, however, commonly used rootfinding schemes might not yield convergent solutions. To clarify this argument, let us rewrite eqn (10) to incorporate nonlocal internal variables

$$f(\mathbf{e}, I_i) = f(\mathbf{e}, I_j^L, I_k^N) \tag{15}$$

in which the total set of internal variables I_i has been decomposed into two parts: local variables I_j^L and nonlocal variables I_k^N, where the subscripts j and k denote the indices of local and nonlocal variables, respectively. The nonlocal variables can be functions of gradients or weighted integral averages of the local variables. Then the use of the damage consistency condition yields

$$\dot{\lambda} = -\frac{\mathbf{e}:\left(\sum_i \mathbf{Q}_i\right):\dot{\mathbf{e}}}{\sum_i \frac{\partial f}{\partial I_i} R_i} = -\frac{\mathbf{e}:\left(\sum_i \mathbf{Q}_i\right):\dot{\mathbf{e}}}{\sum_j \frac{\partial f}{\partial I_j^L} R_j^L + \sum_k \frac{\partial f}{\partial I_k^N} R_k^N}. \tag{16}$$

As can be seen, the explicit formulation of the damage rules, eqn (11), always leads to a closed-form representation for $\dot{\lambda}$ as shown by eqn (16), and for the tangent damage stiffness tensor $\mathbf{T}^{ed}$ which is defined by the following

$$\dot{\mathbf{s}} = \mathbf{T}^{ed}:\dot{\mathbf{e}}. \tag{17}$$

To make this clear, consider the rate form of eqn (7) given by

$$\dot{\mathbf{s}} = \mathbf{D}^{ed}:\dot{\mathbf{e}} + \dot{\mathbf{D}}^{ed}:\mathbf{e}. \tag{18}$$

Then, the use of eqns (9), (11), (16) and (18) yields

$$\begin{aligned} \mathbf{T}^{ed} &= \mathbf{D}^{ed} + \frac{\left(\sum_i R_i \mathbf{Q}_i\right):\mathbf{e}\otimes\mathbf{e}:\left(\sum_i \mathbf{Q}_i\right)}{\sum_i \frac{\partial f}{\partial I_i} R_i} \\ &= \mathbf{D}^{ed} + \frac{\left(\sum_i R_i \mathbf{Q}_i\right):\mathbf{e}\otimes\mathbf{e}:\left(\sum_i \mathbf{Q}_i\right)}{\sum_j \frac{\partial f}{\partial I_j^L} R_j^L + \sum_k \frac{\partial f}{\partial I_k^N} R_k^N}. \end{aligned} \tag{19}$$

As a result, either a secant or tangent damage stiffness tensor can be employed in numerical calculations.

If the nonlocal internal variables appear in the damage surface function, then eqns (16) and (19) involve an integro-differential operator or a higher-order differential operator with inhomogeneous orders, as compared with the case of a local model. Thus, the numerical integration must be simultaneously performed in both time and space to find a nonlocal solution for $\Delta\lambda$, for which special approaches are required (Chen, 1993a; De Borst *et al.*, 1992). Generally, it is not feasible to obtain closed-form solutions for nonlocal models although closed-form representations are available for both $\dot{\lambda}$ and $\mathbf{T}^{ed}$. Nevertheless, by observing carefully the effects of nonlocal terms on the framework of damage models, certain forms of nonlocal models can still be selected so that they can be solved analytically or numerically with high accuracy, as shown next.

3.2. *Formulation with a local damage surface function*

For example, suppose the damage surface function f is constructed so that all the nonlocal terms factor out. Then the damage surface and consistency condition involve local variables only, while the evolution equation for the secant damage stiffness tensor, equation (9), includes the nonlocal variables. Thus, rootfinding schemes used for a local model can be used equally well for a nonlocal model. As an illustration, consider an isotropic nonlocal damage model based on the following choice of functions

$$\mathbf{Q}_1 = 2C_1C_2G^e(1+R_2)\exp[-C_2(I_1+I_2)]\mathbf{P}^d \tag{20a}$$

$$\mathbf{Q}_2 = 0 \tag{20b}$$

$$F^2 = \tfrac{2}{3}C_1C_2G^e(1+R_2)\exp[-C_2(I_1+I_2)][\bar{e}_L(1+I_1)]^2 \tag{20c}$$

$$R_1 = R_1^L = 1 \tag{20d}$$

$$R_2 = R_1^N = \|C_3\nabla I_1\|_2, \tag{20e}$$

where $\mathbf{P}^d$ is defined to be a deviatoric projection tensor such that $\mathbf{e}^d = \mathbf{P}^d:\mathbf{e}$ with $\mathbf{e}^d$ being the deviatoric strain tensor. The positive-definiteness of the tensor $\mathbf{Q}_1$ guarantees that the thermodynamic restrictions are satisfied. The model parameters C_1, C_2, C_3, G^e, and $\bar{e}_L$ are all positive and can be identified from experimental data by direct and indirect comparisons, with G^e being an initial shear modulus and $\bar{e}_L$ corresponding to the uniaxial strain at a limit point. The nonlocal term, R_2, is assumed to be related to a two-norm of the gradient of the local internal variable. Based on eqns (11) and (20), the rate forms of the local and nonlocal internal variables are given by

$$\dot{I}_1 = \dot{I}_1^L = \dot{\lambda} \tag{21a}$$

$$\dot{I}_2 = \dot{I}_1^N = \dot{\lambda}\|C_3\nabla I_1\|_2. \tag{21b}$$

From eqns (9) and (20), it then follows that

$$\dot{\mathbf{D}}^{ed} = -\dot{I}_1\mathbf{Q}_1 = -\dot{\lambda}\{2C_1C_2G^e(1+R_2)\exp[-C_2(I_1+I_2)]\mathbf{P}^d\} \tag{22}$$

which is a nonlocal evolution equation. With $\bar{e} = (\tfrac{2}{3}\mathbf{e}^d:\mathbf{e}^d)^{1/2}$ and due to the common factors between $\mathbf{Q}_1$ and F, the use of eqn (10) results in a local strain-based damage surface

$$f = \bar{e}-\bar{e}_L(1+I_1) = 0. \tag{23}$$

Thus, the damage consistency condition from eqn (23) is the local equation

$$\dot{\lambda} = \dot{I}_1 = \frac{\dot{\bar{e}}}{\bar{e}_L} \tag{24}$$

which is a rather trivial exercise for a numerical algorithm. Preliminary results obtained (Chen and Schreyer, 1990c) indicate that the use of a local damage surface looks quite appealing for computational solutions of localization problems, as further illustrated in Section 6. Conversely, a formulation in which a nonlocal term does not factor out of the damage surface function may lead to a partial differential equation which in general must be solved numerically (De Borst *et al.*, 1992).

Remark 1. When modeling damage effects, it is usually more convenient to formulate first the total form of a secant damage stiffness tensor and then find the corresponding rate form by differentiation in order to check the thermodynamic restrictions based on the basic framework. For example, it is easy to obtain eqn (22) if the following form for $\mathbf{D}^{ed}$ is first postulated

$$\mathbf{D}^{ed} = 3K^e\mathbf{P}^s + 2(G^e + G^d)\mathbf{P}^d \quad \text{with} \quad G^d = C_1G^e\{\exp[-C_2(I_1+I_2)]-1\}, \tag{25}$$

where K^e is an initial bulk modulus, and $\mathbf{P}^s$ is defined to be a spherical projection tensor so that $\mathbf{e}^s = \mathbf{P}^s : \mathbf{e}$ with $\mathbf{e}^s$ being the spherical part of the strain tensor $\mathbf{e}$. Also, it is clear to identify nonlocal degradation features by using the total, instead of the rate, form of a damage tensor, as shown by eqn (25) which reflects the nonlocal damage on the shear modulus.

3.3. *Formulation with analytical solutions*

An alternative approach is to choose f to be a function of a nonlocal variable but to possess a form that allows an analytical solution to the consistency equation. Since it is not generally feasible to solve the highly nonlinear and multi-operator-involved nonlocal constitutive equations, analytical solutions for simple cases are of extreme importance if a constitutive equation solver or a structural solution scheme needs to be verified. In addition, the simple cases for which experimental data are available make clear the physical meaning behind the definition of additional boundary conditions for nonlocal governing equations (Mühlhaus and Aifantis, 1991). To this end, a simple nonlocal damage model is explored based on analytical investigations for nonlocal plasticity and damage (Chen, 1993a; Schreyer, 1989, 1990), which provides another example to show how the specific form of a nonlocal model can be chosen to fit analytical or numerical needs.

In accordance with the basic framework given previously, the following functions are selected

$$\mathbf{Q}_1 = 2(1-I_1)(3K^e\mathbf{P}^s + 2G^e\mathbf{P}^d) \tag{26a}$$

$$\mathbf{Q}_2 = 0 \tag{26b}$$

$$F^2 = 3K^e(1-I_1)\mathbf{e}^s : \mathbf{e}^s + \tfrac{4}{3}G^e\bar{e}_L\frac{(1-I_1-I_2)^2}{(1-I_1)^3} \tag{26c}$$

$$R_1 = R_1^L = 1 \tag{26d}$$

$$R_2 = R_1^N = \frac{2}{\dot{I}_1}\|a\nabla I_1\|_2\|a\nabla\dot{I}_1\|_2. \tag{26e}$$

It should be noted that eqn (26e) requires $\dot{I}_1 > 0$ which is consistent with the evolution of damage. A similar requirement appears in the work by De Borst *et al.* (1992). As compared to eqn (20), the model parameters reduce to a, G^e, and $\bar{e}_L$. The substitution of eqn (26) into eqns (9)–(11) then yields

$$\dot{\mathbf{D}}^{ed} = -\dot{I}\mathbf{Q}_1 = -2\dot{\lambda}(1-I_1)(3K^e\mathbf{P}^s+2G^e\mathbf{P}^d) \tag{27a}$$

$$f = \bar{e}-\bar{e}_L\frac{1-I_1-I_2}{(1-I_1)^2} \tag{27b}$$

$$\dot{I}_1 = \dot{I}_1^L = \dot{\lambda} \tag{27c}$$

$$\dot{I}_2 = \dot{I}_1^N = 2\|a\nabla I_1\|_2\|a\nabla\dot{I}_1\|_2. \tag{27d}$$

The total forms corresponding to the above rate forms are given by

$$\mathbf{D}^{ed} = (1-I_1)^2(3K^e\mathbf{P}^s+2G^e\mathbf{P}^d) \tag{28a}$$

$$I_1 = I_1^L = \lambda \tag{28b}$$

$$I_2 = I_1^N = \|a\nabla I_1\|_2^2. \tag{28c}$$

As can be seen from eqns (27) and (28), the damage stiffness tensor represents a conventional scalar damage law which is a local formulation while the strain-based damage surface involves a nonlocal internal variable. The definition of the nonlocal variable can be easily identified from the total form, eqn (28c). Although the nonlocal damage surface results in a nonlocal consistency condition, analytical solutions can be obtained for the given model in one-dimensional cases. Since three-dimensional failure is frequently manifested as a planar band, the use of a normal coordinate implies that analytical solutions for three-dimensional cases may also be obtained.

For the purpose of illustration, consider a homogeneous bar of length L loaded in tension, as shown in Fig. 1. The nonlocal damage model defined by eqns (27) and (28) reduces in this case to

$$f = e-e_L\frac{1-I_1-I_2}{(1-I_1)^2} \tag{29a}$$

$$s = E(1-I_1)^2 e \tag{29b}$$

$$I_1 = \lambda \tag{29c}$$

$$I_2 = (aI_{1,x})^2 \tag{29d}$$

in which e and s are uniaxial tensile strain and stress, respectively, and E denotes Young's modulus. If $a = 0$, the nonlocal model becomes local. When damage occurs, it follows from eqns (29a) and (29b) that

$$s = Ee_L(1-I_1-I_2) = s_L(1-I_1-I_2), \tag{30}$$

where s_L denotes the limit stress. Suppose the origin, $x = 0$, is chosen to be the point at which an initial imperfection is introduced. The use of the equilibrium condition, $s_{,x} = 0$, and eqn (30) then yields analytically the distribution of I_1 and e for the conventional boundary conditions and an additional symmetry boundary condition, $I_{1,x}(0) = 0$. The solutions by Chen (1993a) for the current case can be written as

$$I_1 = \begin{cases} \lambda = 1-\dfrac{s}{s_L}-\left(\dfrac{x}{2a}\right)^2, & 0 \leqslant x \leqslant x^* \\ 0, & x^* \leqslant x \leqslant L \end{cases} \tag{31}$$

$$e = \begin{cases} \dfrac{s}{E\left[\dfrac{s}{s_L}+\left(\dfrac{x}{2a}\right)^2\right]^2}, & 0 \leqslant x \leqslant x^* \\ \dfrac{s}{E}, & x^* \leqslant x \leqslant L \end{cases} \tag{32}$$

in which x^* is the boundary between the localized and nonlocalized deformation fields defined by $I_1(x^*) = 0$. As can be seen from eqn (31), the damage internal variable is symmetrical with respect to $x = 0$, which is usually the case for a center-notched specimen under tensile loading if $x = 0$ is considered as the center. Thus, the selection of a symmetry boundary condition for the nonlocal governing equation might be appropriate based on the experimental studies of material and interface failures (Wang and Maji, 1993).

Remark 2. Since $I_1(x^*) = 0$, it follows from eqn (31) that

$$x^* = 2a\left(1 - \frac{s}{s_L}\right)^{1/2} \tag{33}$$

which implies that the boundary between the localized and nonlocalized deformation fields is moving with a decrease in stress but with an increase in damage. Based on the concept of a moving boundary, another simple approach can be taken to solve for localization problems (Chen, 1993b). When the load-carrying capacity of the bar reaches zero, the boundary point, x^*, achieves its maximum value, $x^* = 2a$. Since the final width of a localization zone can presumably be measured experimentally, the model parameter a can be identified.

With the use of eqn (32), the total deflection over the bar, δ, can be found as

$$\delta = \int_0^{x^*} \frac{s}{E\left[\frac{s}{s_L} + \left(\frac{x}{2a}\right)^2\right]^2} \mathrm{d}x + \int_{x^*}^{L} \frac{s}{E} \mathrm{d}x. \tag{34}$$

Thus, a constitutive equation solver or a structural solution scheme can be verified in one-dimensional cases. A generalization of the one-dimensional analytical solutions to three-dimensional problems might be available provided that a semi-analytical solution procedure is employed (Chen, 1993a). Hence, it might not be necessary to perform the numerical integration in both space and time to find a nonlocal solution for $\Delta\lambda$, although the consistency condition is nonlocal for this nonlocal damage model.

In summary, the essence of formulating a solvable nonlocal damage model is to choose suitable damage surfaces so that the model parameters can be determined experimentally and the evolution of the local internal variable can be found analytically or numerically with the required accuracy. Thermodynamic restrictions can be satisfied if the basic framework of damage models is followed.

4. A ROBUST SOLUTION SCHEME FOR FAILURE PREDICTION

As can be observed from the above basic framework, the constitutive-model-solvers used for local models (Chen and Schreyer, 1990b, 1991) can be applied to nonlocal counterparts with minor modifications if the consistency conditions remain local. Especially for strain-based damage models such as that defined by eqn (20), the algorithm used for local formulations can be equally well employed for nonlocal ones. The nonlocal internal variable is calculated on the structural level within an iterative loop so that a unique solution is obtained after the convergence criterion is satisfied. For a dynamic problem, a suitable time integrator can be employed to obtain numerical solutions of nonlinear structural response. When the external loading is static, a well-conditioned incremental–iterative scheme must be developed together with a reasonable constraint in order to obtain the post-limit solutions including snap-back and snap-through phenomena. This section is mainly devoted to static analyses for which severe numerical difficulties are more prevalent.

An extensive investigation has been conducted to select a suitable constraint for predicting structural failure, as briefly reviewed in Section 1. Since large deformation is localized into small region during the damaging process, a suitable constraint for localization problems must be formulated in terms of a localized kinematical field. Experimental studies

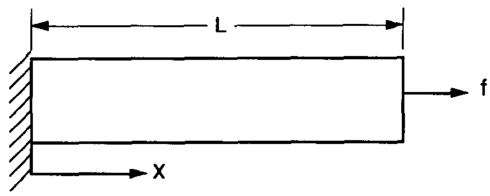


Fig. 1. Geometry and notation of a one-dimensional model problem.

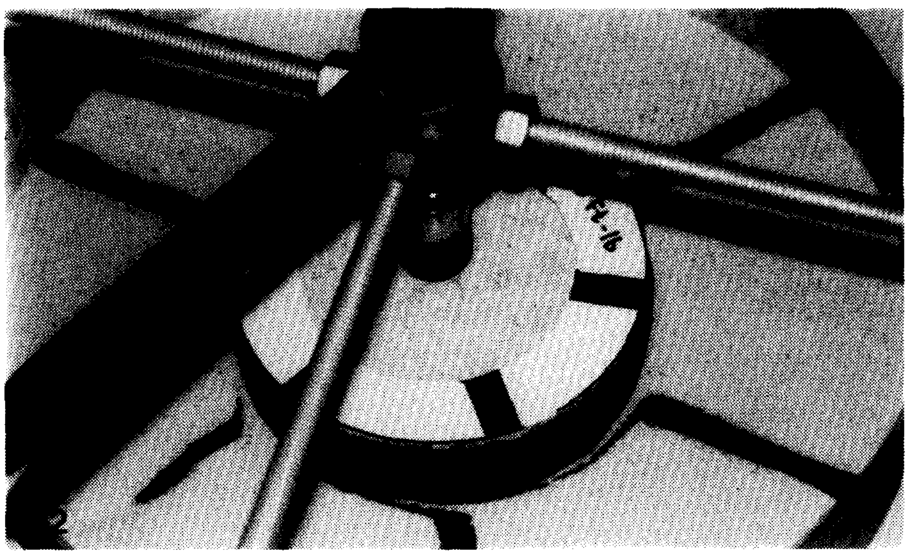


Fig. 2. Post-test configuration of circular soil–concrete interface (Chen, 1989).

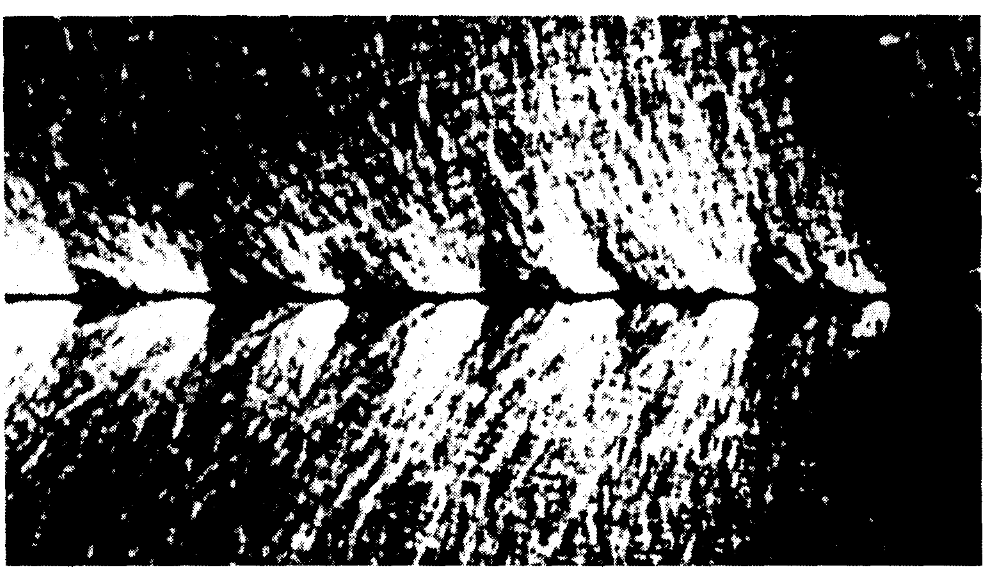


Fig. 3. Localization near geogrid–soil interface (Carroll and Chouery-Curtis, 1990).

also indicate that a localized kinematical variable, instead of load or total displacement, must be invoked in a closed-loop servo-system to obtain experimental data of the post-limit response associated with snap-back or snap-through. Due to the fact that the localization zone is evolving and there exists a sign change of the load increment at a critical point, the control element should vary in position based on a suitable measure of damage, and the control parameter should be monotonically increasing which reflects the extent of irreversible damage (Chen and Schreyer, 1990a, 1991). In other words, the control condition imposed on the incremental–iterative scheme for static damage analysis must reflect the physical nature of the damage evolution. Preliminary results obtained for one- and two-dimensional sample problems indicate that the evolving-localization control is a reasonable choice for localization problems including snap-back or snap-through. Because the evolving-localization control is related to the kinematical field, this procedure can also be valid for geometrically nonlinear problems such as buckling.

For materially nonlinear problems, a differential (rate) form of constitutive models is preferred in order to trace those paths that are history dependent. Hence, a tangent stiffness tensor can be directly derived from the differential form of the stress–strain relation, and a resulting tangent stiffness matrix is commonly employed in an incremental–iterative structural solution scheme. When the tangent stiffness matrix becomes ill-conditioned around a critical (bifurcation or limit) point, the numerical solution will converge very slowly or even fail at some point on the solution path, which is particularly true for a smooth stress–strain relation. As a remedy to circumvent the occurrence of an ill-conditioned tangent matrix around critical points, a secant matrix based on damage mechanics has been used together with an evolving-localization constraint (Chen and Schreyer, 1991). As a result, the rate of convergence is improved dramatically. Although there exists a one-to-one correspondence between secant and tangent stiffness tensors for the differential form of damage models, it is not natural to derive a secant stiffness tensor for existing plasticity models due to the permanent deformation observed in the plastic process (Chen and Schreyer, 1990b).

In order to incorporate both damage and plasticity models into one computer code, it is proposed that an incremental–iterative solution scheme be constructed through the use of an initial elasticity stiffness matrix together with an evolving-localization constraint. Thus, only one inverse calculation is required, and the amount of computation involved in the iterative loop is dramatically reduced. However, it should be emphasized that the use of an initial elasticity stiffness matrix does not exclude the existence of critical points that can be determined only by analyzing the corresponding tangent stiffness matrix. For example, there might be more than one bifurcation point at the same time or a second bifurcation point might follow the occurrence of the first one. The procedure for tracing the post-bifurcation path depends mainly on the choice of a suitable constraint instead of a stiffness matrix. Different constraints generally generate different solution paths. In other words, the constraint imposed on the solution path plays a crucial role whether or not the stiffness matrix is ill-conditioned. Thus, the dependence of the evolving-localization constraint on the location of initial imperfections and on the evolution history of localization detects the solution path following the critical point, while the use of a well-conditioned stiffness matrix guarantees that a numerical solution can be obtained. Hence, the solution scheme proposed here yields a unique combination of a suitable constraint and well-conditionness, and reflects the physical mechanisms of structural failure that are quite sensitive to initial imperfections and boundary conditions.

Within the context of proportional loading, the external load vector can be represented by $\mu\{\mathbf{q}^*\}$ in which μ is the magnitude of the load and $\{\mathbf{q}^*\}$ the reference load vector. With the load level at the end of the previous incremental step denoted as $\mu_0\{\mathbf{q}^*\}$, the proposed iterative loop for a given loading increment is outlined as follows:

$$\{\mathbf{p}\}_{i-1} = \mu_0\{\mathbf{q}^*\} - \int_V [\mathbf{B}]^T\{\mathbf{s}\}_{i-1}\,\mathrm{d}V \tag{35a}$$

$$\{\delta\mathbf{a}\}_i = [\mathbf{K}]^{-1}(\Delta\mu_i\{\mathbf{q}^*\} + \{\mathbf{p}\}_{i-1}) = \Delta\mu_i\{\delta\mathbf{a}^q\} + \{\delta\mathbf{a}^p\}_i \tag{35b}$$

$$\{\Delta\mathbf{a}\}_i = \{\Delta\mathbf{a}\}_{i-1} + \{\delta\mathbf{a}\}_i, \tag{35c}$$

where the subscript i is the iteration index. The matrix **[B]** relates the strain and nodal displacement components through a differential operator and interpolation functions, and $\{\mathbf{p}\}$ is the out-of-balance force vector. To find the load increment $\Delta\mu_i$ indirectly through the evolving-localization constraint, the displacement increment $\{\delta\mathbf{a}\}$ is divided into two parts: $\{\delta\mathbf{a}^q\}$ and $\{\delta\mathbf{a}^p\}$. As can be seen, an LU decomposition on the initial elasticity stiffness matrix, **[K]**, needs to be performed only for the first iteration in the first incremental step. Thus, $\{\delta\mathbf{a}^q\}$ can be fixed during the iteration. The evolving-localization control is imposed on the incremental displacement vector and takes the following form

$$\langle\mathbf{c}\rangle\{\Delta\mathbf{a}\}_i = \Delta\eta, \tag{36}$$

where the localized constraint vector $\{\mathbf{c}\}$ is formulated in terms of a control element and varies in position according to the measure of damage rate. Other control methods such as direct- or indirect-displacement control are special cases of eqn (36). The sign of the constraint parameter $\Delta\eta$ is chosen to be positive for loading, which represents the irreversible evolution of damage. Since the convergence criterion must be satisfied at the end of the previous increment and $\Delta\eta$ is to be fixed through the load step, then the use of eqns (35) and (36) yields the following expression for the external load increment, $\Delta\mu_i$

$$\Delta\mu_i = \frac{\Delta\eta}{\langle\mathbf{c}\rangle\{\delta\mathbf{a}^q\}}, \qquad i = 1 \tag{37a}$$

$$\Delta\mu_i = -\frac{\langle\mathbf{c}\rangle\{\delta\mathbf{a}^p\}_i}{\langle\mathbf{c}\rangle\{\delta\mathbf{a}^q\}}, \quad i > 1. \tag{37b}$$

Hence, the sign change of the external load increment around a critical point is determined by the evolving-localization control. The total strain increment is calculated from the incremental displacement vector, $\{\Delta\mathbf{a}\}_i$, and then is used to update the stress $\{\mathbf{s}\}_i$ based on a constitutive equation solver.

The other details of the incremental–iterative solution scheme, including the selection of a suitable convergence criterion for localization problems, have been given previously (Chen and Schreyer, 1990a, 1991). The major change in this paper is the use of an initial elasticity stiffness matrix instead of the secant or tangent stiffness for both nonlocal damage and plasticity. In order to illustrate the proposed procedure, the prediction of interface failure is considered next.

5. INTERFACE PHENOMENA AND MODELING

Common experimental methods for determining interface properties include direct shear test, torsional test, pullout test and triaxial test, for both quasi-static and dynamic loadings. Recently, research emphasis has been on understanding the failure mechanisms of interfaces via micromechanical techniques. Both phenomenological and micromechanical data have been available for certain interfaces consisting of geologic materials, and these results indicate the need for simulating nonlocal failures adjacent to an interface.

Torsional tests of soil–concrete interfaces were performed with photographs taken to show the changing deformation field adjacent to the interface (Chen, 1989). In the simple experimental device, a solid concrete cylinder was placed in the center of a large cylinder with the space between the cylinders filled with sand. Cardboard forms in the radial direction were located at 60° intervals. Black sand was placed in the forms at the same rate as uncolored sand was rained into the space between the forms. The forms were pulled up as the sand was placed with the result that radial markers were placed in the sand for the complete axial distance of the container. The sand markers were aligned with black stripes painted on the top surface of the concrete cylinder, which also contained an embedded bolt through which an external torque could be applied. When the concrete cylinder is rotated relative to the sand container, the required torque rises and then falls to a residual value with the evolution of a shear band adjacent to the interface. This reduction in torque indicates that a pronounced softening effect is present. No apparent slip occurs until the evolution of the shear band stops. The post-test configuration of the deformation field near a circular soil–concrete interface is shown in Fig. 2. In the study of geogrid connections (Carroll and Chouery-Curtis, 1990), the evolution of a localized deformation zone, as shown in Fig. 3, can also be observed adjacent to the geogrid–soil interface.

Although a number of factors not directly pertinent to the interface problem might act to obscure the true phenomenon, it is still possible, based on a literature review (Chen, 1993a; Yuan, 1990), to describe a large class of interface problems as follows.

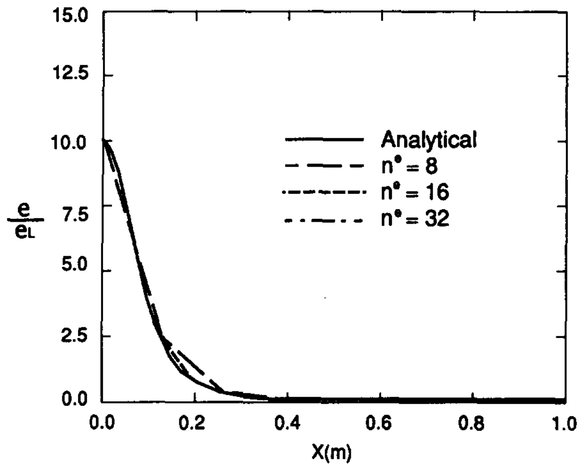
The interface usually initiates progressively distributed damage such as dispersed microcracks, void formation or loss of interparticle contacts in one or both of the adjacent materials. From a macroscopic, or continuum, point of view, large deformations can be observed adjacent to the interface. The deformation field in the softer of the two materials at contact starts to change significantly around the peak load, and is then localized into a zone which evolves as damage or apparent softening progresses. It appears that no apparent slip occurs in the post-limit regime until the evolution of the localization zone stops. When macroscopic slip occurs at the interface, the deformation field of the softer material is no longer compatible with the stiffer one, and the size of the localization zone is usually fixed and looks different for different materials. Thus, nonslip, damage, and localization seem to be nontrivial features of interface phenomena, especially during the post-limit regime. Hence, a suitable constitutive theory is required to predict interface failures with correct energy dissipation.

In order to predict the observed phenomena of interface failures, it was suggested that a nonlocal plasticity or damage model be used to simulate the softer of two materials at contact with the assumption that the interface, in effect, weakens the softer material due to the introduction of microcracks and microvoids (Chen and Schreyer, 1987; Chen, 1989, 1993a). If an interface can be dealt with in such a manner, then special interface elements with a fictitious dimension is not needed for numerical simulation, and furthermore, the evolution of a localized deformation zone adjacent to the interface can be predicted that is representative of physical mechanisms. For the purpose of establishing a sound foundation for the use of nonlocal models in assessing interface failures, the proposed solution strategy is first verified by comparing numerical solutions with analytical solutions in one-dimension, and then applied to the simulation of a two-dimensional pullout test.

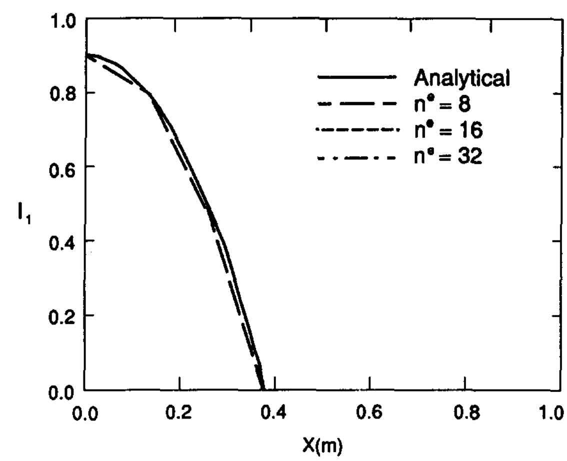
6. NUMERICAL RESULTS

One-dimensional analytical solutions are available for both nonlocal damage and plasticity models (Chen, 1993a; Schreyer, 1989, 1990) which qualitatively reflect the observed features of localization problems. The case with the use of a simple nonlocal damage model as summarized by eqn (29) is considered here for the purpose of illustration. Thus, a semi-analytical procedure can be employed to solve the model. Similar results hold for nonlocal plasticity models.

For the static bar depicted in Fig. 1, a uniform mesh consisting of constant stress elements is used for the semi-discretization process, with bar length $L = 1$ m and zone width $2a = 0.4$ m. The model parameters are assigned with the following values that are of the correct order of magnitude for concrete materials under tension: $E = 35$ GPa and $s_L = 3.5$ MPa. In order to trigger a localized deformation mode, the first element is chosen as an initial imperfection with a weak limit $s_L^w = 0.99 s_L$. The local damage variable is determined via an analytical solution, while the gradient of damage is evaluated numerically through a simple difference with respect to two neighboring elements. The gradient in the weak element is zero because of the symmetric distribution of damage. If a dynamic case is of concern, special care must be taken to avoid spurious diffusion and oscillations due to discretization procedures (Chen and Schreyer, 1987; Lasry and Belytschko, 1988). In numerical calculations, the nonlocal damage relation is used for all the elements inside the localization zone while the elasticity relation holds outside the zone with the boundary of the zone moving according to the nonlocal criterion. The analytical and numerical solutions



(a)



(b)

Fig. 4. Comparisons of analytical and numerical solutions. (a) Strain distribution and (b) damage distribution.

are compared in Figs 4(a) and 4(b) for the normalized strain and damage distributions along the bar at stress level $s/s_L = 0.1$. As can be seen, the numerical solutions converge very quickly to the analytical solution as the number of finite elements, n^e, increases. In addition, a localization zone of finite width is obtained. The width does not decrease when the mesh is refined, as would be the case for a local constitutive model. Different material usually yield localization zones of different shapes and sizes that should be identified based on micromechanical experiments.

For general cases, analytical solutions are usually not available. An important problem in geotechnical engineering is the pullout test, as shown in Fig. 5, which is used to illustrate the proposed numerical solution procedure. The isotropic nonlocal damage model defined by eqn (20) is adopted to catch the essential features of nonlocal failure as observed adjacent to the geogrid–soil interface. As a result, the consistency condition is local, and the algorithm used for local models can be equally well used here. The nonlocal internal variable is determined on the structural level within an iterative loop. As can be found easily from eqn (25), the secant damage stiffness tensor simulates the nonlocal shear degradation mode. Figure 6 illustrates the local features of the nonlocal model with the values of model parameters being representative of soil materials under shear. As can be seen, no permanent

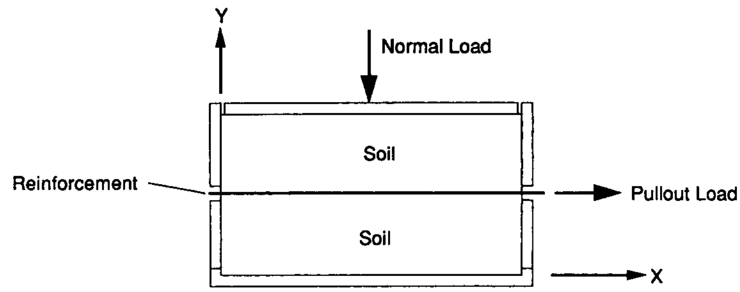


Fig. 5. Geometry and notation of a pullout test.

strain occurs upon unloading. If C_3 in eqn (20e) is nonzero, the nonlocal term is activated. Due to the symmetry of the pullout test, the lower half of the problem domain, defined by $0 \leqslant X \leqslant 1$ m and $0 \leqslant Y \leqslant 1$ m, is considered in numerical illustrations. The effect of pullout load at the interface, $Y = 1$ m, is simulated by prescribing tangential displacements along that interface with the exception that the two corner nodes are made traction-free to avoid the singularities. The bottom boundary at $Y = 0$ is fixed, and the left boundary at $X = 0$ and right boundary at $X = 1$ m are traction-free. The mesh consists of rectangular elements each of which is constructed from four constant stress triangles. The gradient of the damage measure is evaluated only along the Y-axis by a simple difference with the X-components being ignored. A zero-gradient boundary condition is assumed at $Y = 1$ m. Final deformation patterns next to the geogrid–soil interface are shown in Fig. 7(a) with $C_3 = 0$ and 7(b) with $C_3 = 0.02$ m by using a coarse mesh. The solutions are replicated for a fine mesh and shown in Fig. 7(c) with $C_3 = 0$ and Fig. 7(d) with $C_3 = 0.02$ m. Note that for the local model with $C_3 = 0$ deformation always localizes into one row of elements, while localization occurs in a zone governed by the material model and not the mesh when $C_3 \neq 0$. The corresponding load–displacement curves are given in Fig. 8(a) with $C_3 = 0$ and Fig. 8(b) with $C_3 = 0.02$ m. As can be found from these figures, experimental observations of the pullout test are qualitatively predicted with the use of the proposed procedure, and the nonlocal effect of the damage model reduces the mesh-dependence of numerical solutions for localization problems. It must be emphasized, however, that small deformations have been assumed here even though the deformations in the localization zone are usually large. Hence, the analysis can be expected to only provide a qualitative comparison with experimental data. The torsional experiment, as shown in Fig. 2, has not been analyzed here because the corresponding numerical model is basically one-dimensional.

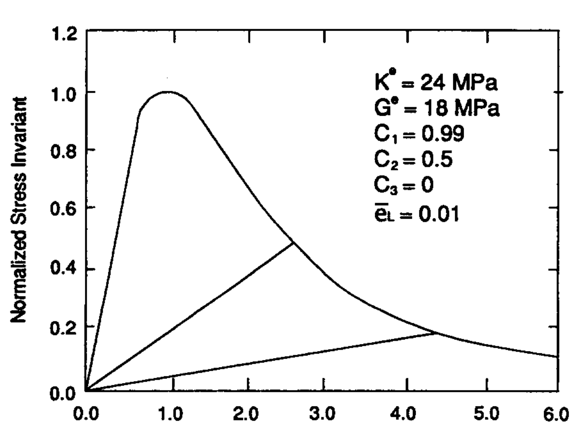


Fig. 6. Loading and unloading of the nonlocal damage model.

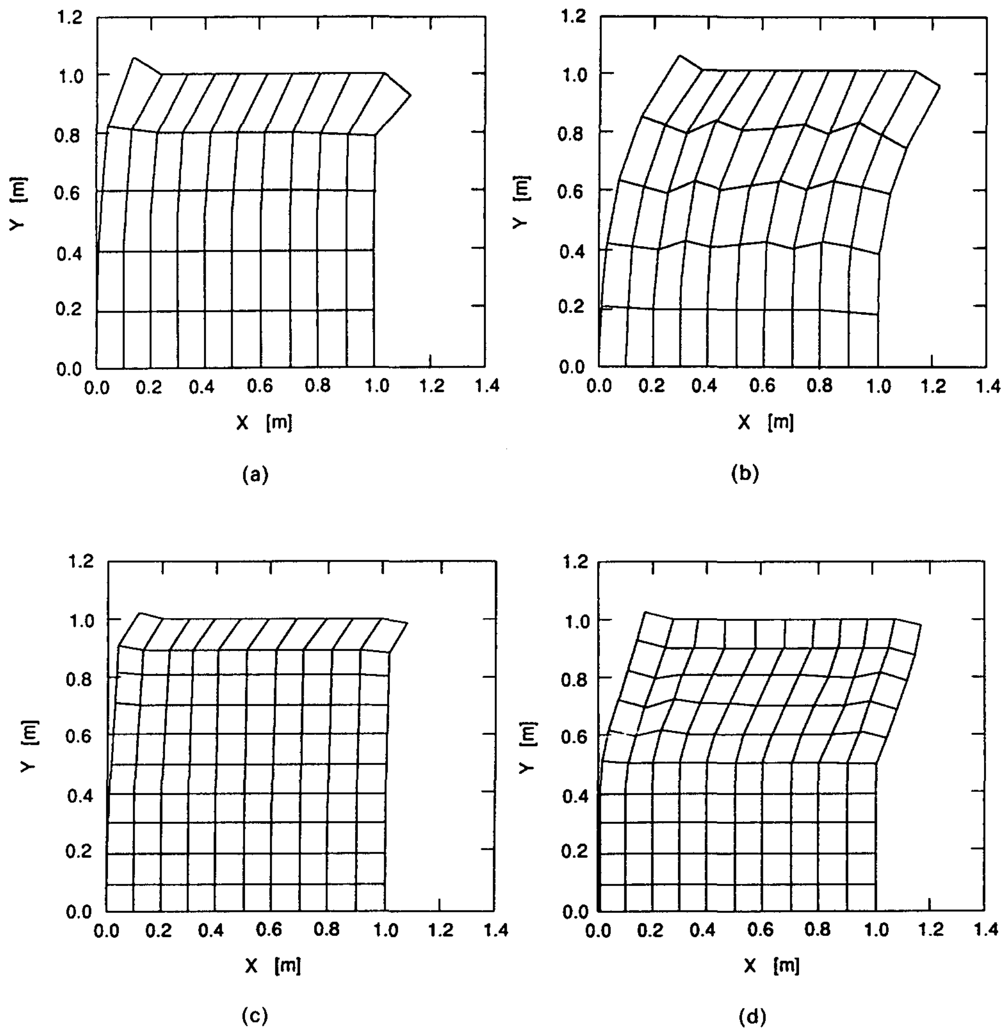
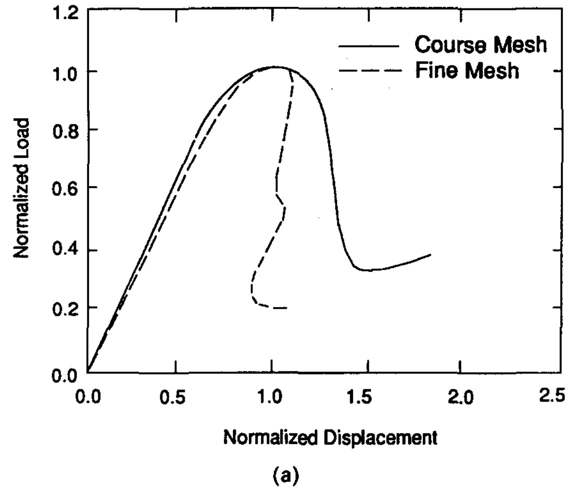


Fig. 7. Comparisons of deformation patterns between local and nonlocal models. (a) Local model with coarse mesh, (b) nonlocal model with coarse mesh, (c) local model with fine mesh, and (d) nonlocal model with fine mesh.

7. DISCUSSION

A basic framework of nonlocal damage models is established based on thermodynamic restrictions. The proposed solution procedure, involving a combination of an initial elasticity stiffness matrix and an evolving-localization constraint, is verified through comparisons with analytical solutions and experimental observations. The damage evolution next to an interface is qualitatively predicted with a simple nonlocal damage model. Although sample problems considered here are one-dimensional in nature, this kind of problem would be important if one wished to identify material properties.

Several alternatives exist for the nonlocal formulation of constitutive models, each of which has an appealing feature for specific cases. The way to evaluate nonlocal internal variables is a crucial factor in selecting a suitable model for application. Since nonlocal features of material behavior can be observed not only in solids but also in fluids (Chen and Clark, 1991), the results presented here should be of value for most engineering materials. It appears that a nonlocal approach together with a statistical specification of material properties should be considered if a general case of failure is of concern. Because failure phenomena usually occur at a highly localized part of the entire structure, it becomes



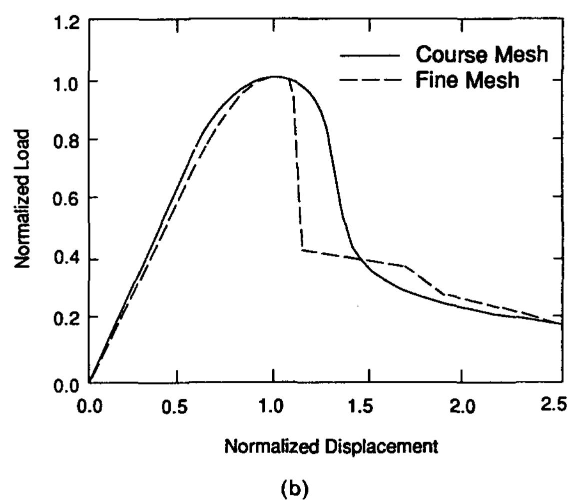


Fig. 8. Convergence study of local and nonlocal damage models with different meshes. (a) Local damage model and (b) nonlocal damage model.

more troublesome to apply a single nonlocal model to the whole problem domain if the details of failure need to be predicted with a limited computer capacity. As an alternative, a partitioned-solution method has been proposed for nonlocal plasticity (Chen, 1993b) and nonlocal creep damage (Chen and Wang, 1993). The key idea is to treat a complicated problem with two simple problems that are governed by different constitutive laws and by moving boundaries inside the problem domain, as observed in experiments for localization problems and illustrated by the numerical results of Section 6. Since localized deformations are usually large, the small-deformation assumption might be valid only into the immediate post-limit regime. A possible extension of the current work to a case of localized large deformations is to adopt a particle scheme (Sulsky *et al.*, 1992) or its alternative (Belytschko *et al.*, 1993). As a result, a few material points might be enough to resolve the localized damage zone with correct energy dissipation. With the use of an energy-based criterion to identify the transition between volume cracking and surface cracking (Xie *et al.*, 1994), a complete failure process, including diffused, localized and discrete mechanisms might then be described within a unified framework.

In fact, the essence of various nonlocal approaches or localization limiters is nothing but controlling the evolution of inhomogeneous interactions among material points. Hence, future research on experimental, analytical and numerical aspects of localization problems should be able to find an approach of practical value in a general case if the essential feature of the evolution process is well understood.

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